

COHOMOLOGY OF CONGRUENCE SUBGROUPS OF $\mathrm{SL}_4(\mathbb{Z})$. III

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ABSTRACT. In two previous papers [AGM02, AGM08] we computed cohomology groups $H^5(\Gamma_0(N); \mathbb{C})$ for a range of levels N , where $\Gamma_0(N)$ is the congruence subgroup of $\mathrm{SL}_4(\mathbb{Z})$ consisting of all matrices with bottom row congruent to $(0, 0, 0, *) \pmod{N}$. In this note we update this earlier work by carrying it out for prime levels up to $N = 211$. This requires new methods in sparse matrix reduction, which are the main focus of the paper. Our computations involve matrices with up to 20 million nonzero entries. We also make two conjectures concerning the contributions to $H^5(\Gamma_0(N); \mathbb{C})$ for N prime coming from Eisenstein series and Siegel modular forms.

1. INTRODUCTION

In two previous papers [AGM02, AGM08] we computed the cohomology in degree 5 of congruence subgroups $\Gamma_0(N) \subset \mathrm{SL}_4(\mathbb{Z})$ with trivial \mathbb{C} coefficients, where $\Gamma_0(N)$ is the subgroup of $\mathrm{SL}_4(\mathbb{Z})$ consisting of all matrices with bottom row congruent to $(0, 0, 0, *) \pmod{N}$. The highest level we reached was $N = 83$. We also computed some Hecke operators on these cohomology groups and identified the cohomology as either cuspidal or as coming from the boundary (Eisensteinian).

In this paper we concentrate on explaining new techniques we have developed to reduce very large sparse matrices. These techniques have enabled us to carry out our computations for much higher values of the level N . We explain in Section 2 that our algorithms differ from others in the literature because we must compute change of basis matrices. As an oversimplified illustration, imagine solving $A\mathbf{x} = \mathbf{b}$ for an invertible matrix A . Classical dense methods produce an invertible change of basis matrix P where PA has a desirable form, and we solve for \mathbf{x} by computing $P\mathbf{b}$. When A is large and sparse, computing P is much too expensive if finding \mathbf{x} is our only goal. Iterative methods like Wiedemann's produce \mathbf{x} more simply. In this paper, however, we compute explicit cocycles in our cohomology groups, and compute their images under Hecke operators. As explained in (2.2), change of basis matrices are essential for this task. (See Section 2 for references. The illustration is oversimplified because the actual A have less than full rank and are not even square.)

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The linear algebra issues are compounded when computing the Smith normal form (SNF) of integer matrices. Modern SNF methods reduce mod p for a range of p , solve at each p by iterative methods, and lift to a final result over \mathbb{Z} by Chinese remaindering techniques. It seems unknown, though, how to find the SNF change of basis matrices by Chinese remaindering. Hence we use a different approach (2.1). See (2.3.6) for a comparison of times. Although iterative methods can be efficiently parallelized, this paper does not use parallel techniques.

In future installments of this project we will look at torsion classes in $H^5(\Gamma_0(N); \mathbb{Z})$ as well as twisted coefficient modules. For the torsion classes, we will test Conjecture B of [Ash92] that asserts that they have attached Galois representations. The new sparse matrix techniques discussed here will be of great importance in carrying this project forward.

In the paper [vGvdKTV97] of Van Geemen, Van der Kallen, Top, and Verberkmoes, cohomology classes for $GL(3)$ were found by working modulo small primes and using the LLL algorithm to reconstruct integral solutions. This is a useful method that various people (including ourselves in the past) have followed. However, in this paper we work solely modulo a 5 digit prime p without lifting to \mathbb{Z} . Lifting to \mathbb{Z} would be a huge computational effort at larger levels. The prime p is small enough to make computation fast, and large enough to make us morally certain that we are actually finding the complex betti numbers and Hecke eigenvalues. The fact that all our data is accounted for by Eisenstein series and liftings of automorphic forms confirms this.

We continue to find that the cuspidal part consists of functorial lifts of Siegel modular forms from paramodular subgroups of $Sp_4(\mathbb{Q})$ that are not Gritsenko lifts, as described in [AGM08] for levels $N = 61, 73, 79$. We conjecture that these functorial lifts will always occur, at least for prime level, in Conjecture 2 of Section 4. These lifted forms correspond to selfdual automorphic representations on $GL(4)/\mathbb{Q}$. We were hoping to find non-lifted cuspidal cohomology classes, which would correspond to non-selfdual automorphic representations. Unfortunately, we found none. We see no reason why they should not exist for larger N , but no one has proven their existence. It should be noted that non-selfdual Galois representations, that by Langlands' philosophy would predict non-selfdual automorphic representations for $GL(4)$ of the type we are searching for, were constructed by J. Scholten [Sch02].

Our data for the boundary cohomology for prime level leads to our Conjecture 1 of Section 4 that identifies its constituents as various Eisensteinian lifts of certain classical cuspforms of weights 2 and 4, and of certain cuspidal cohomology classes for $GL(3)/\mathbb{Q}$. This conjecture is a refinement of Conjecture 1 of [AGM08].

We ought to have similar conjectures for composite level, but we don't have enough data to justify an attempt to make them. The size of the matrices and the complexity of computing the Hecke operators increases as N grows larger or more composite. Therefore at a certain point we stopped computing for composite N but were able to continue for prime N up to level 211. Similarly the size of the computation of the Hecke operators at a prime l increases dramatically with l , so that in fact for the new levels in this paper, we compute the Hecke operators at most for $l = 2$.

The index of $\Gamma_0(N)$ in $SL_4(\mathbb{Z})$ grows like $O(N^3)$. Thus the matrices we need to reduce are on the order of $N^3 \times N^3$. This growth in size is what makes this computational project so much more difficult to carry out for large N , compared to the

cohomology of congruence subgroups of $SL_2(\mathbb{Z})$, $SL_3(\mathbb{Z})$, $Sp_4(\mathbb{Z})$, and other lower rank groups. The implied constant in the $O(N^3)$ is larger when N is composite, which is why we eventually had to restrict ourselves to prime N . Also the Hecke operator computations become much more lengthy for composite N .

Please refer to [AGM02] for explanations of why we look in degree 5, the decomposition of the cohomology into the cuspidal part and the part coming from the Borel–Serre boundary, and how we perform the computations. We also review there the number theoretic reasons for being interested in this cohomology, primarily because of connections to Galois representations and motives. In [AGM08] the reader will find how we identified the cuspidal part of the cohomology as lifted from $GSp(4)/\mathbb{Q}$ and why this must be so for selfdual classes of prime level.

In [AGM02] we explained that the well-rounded retract W of the symmetric space for $SL_4(\mathbb{R})$ is a contractible cell complex on which $SL_4(\mathbb{Z})$ acts cellularly with finite stabilizers of orders divisible only by 2, 3 and 5, and that W has only finitely many cells modulo $SL_4(\mathbb{Z})$. Therefore we can use the chain complex $C^*(W/\Gamma_0(N); \mathbb{F})$ to compute $H^*(\Gamma_0(N); \mathbb{F})$ for any field \mathbb{F} of characteristic not equal to 2, 3 or 5. In practice we substitute a large finite field for \mathbb{C} as justified in [AGM02].

Also in [AGM02] we described explicitly and in detail how to handle the data structures needed to construct the chain complex from $W/\Gamma_0(N)$ and hence to create the matrices whose nullspaces modulo column spaces are isomorphic to the cohomology.

In this paper we continue to use this set-up. The new thing here is the method explained in Section 2, which enables us to take N as far as 211.

In Section 3 we give the background on Eisenstein cohomology and Siegel modular forms needed to present our computational results and to formulate our conjectures. Finally, in Section 4 we state two conjectures about the structure of $H^5(\Gamma_0(N); \mathbb{C})$ for N prime, give the results of our computations of $H^5(\Gamma_0(N); \mathbb{C})$ for N prime, $83 \leq N \leq 211$, and verify the two conjectures in this range. The first, Conjecture 1, improves on [AGM02, Conjecture 1] by fixing the weight 4 part of the Eisensteinian cohomology to those weight 4 cuspforms f whose central special value vanishes. We also feel confident now of conjecturing that our list of classes for the boundary cohomology is complete in this case. The second, Conjecture 2, states exactly which cuspidal paramodular Siegel forms at prime level show up in the cuspidal cohomology.

2. COMPUTATIONAL METHODS

Our problem is to find H^5 of a complex of free R -modules for some ring R ,

$$(1) \quad 0 \longleftarrow C^6 \xleftarrow{d^5} C^5 \xleftarrow{d^4} C^4 \longleftarrow \dots$$

Let $n_i = \text{rank } C^i$. View the C^i as a space of column vectors, and represent the d^i as matrices. All the matrix entries lie in \mathbb{Z} . It is possible to carry out our computations over $R = \mathbb{Z}$, obtaining the torsion in the cohomology along with the free part. Our next paper (IV in this series) will study the torsion. In the present paper, we work over a field. However, we will intersperse remarks on the computational problem for more general R , with an eye to future papers in the series.

In principle we want to work over $R = \mathbb{C}$, because our purpose is to study automorphic forms. To avoid round-off error and numerical instability, however, we replace \mathbb{C} with a finite field \mathbb{F}_p of prime order p . If p is large enough, it is

extremely likely that $\dim H^i(\Gamma_0(N); \mathbb{F}_p)$ will equal $\dim H^i(\Gamma_0(N); \mathbb{C})$ for the N we consider, and that Hecke computations will work compatibly. We generally use $p = 12379$, the fourth prime after 12345. We give details about the choice of p in (2.3.3).

The matrices are sparse matrices, meaning only a small fraction of the entries in each row and column are nonzero. Our largest matrices, those for d^4 , have dimension $n_5 \times n_4 \approx N^3/10 \times 25N^3/72$ for prime N . However, at most 6 of the entries in each column are nonzero, and at most 26 in each row. The 6 and 26 are independent of N . The matrices for d^5 have dimension $n_6 \times n_5 \approx N^3/96 \times N^3/10$ for prime N . All these estimates are a few times larger for composite N . We give more precise estimates for the n_i in (2.1.4). The relative sizes are shown in Figure 1.

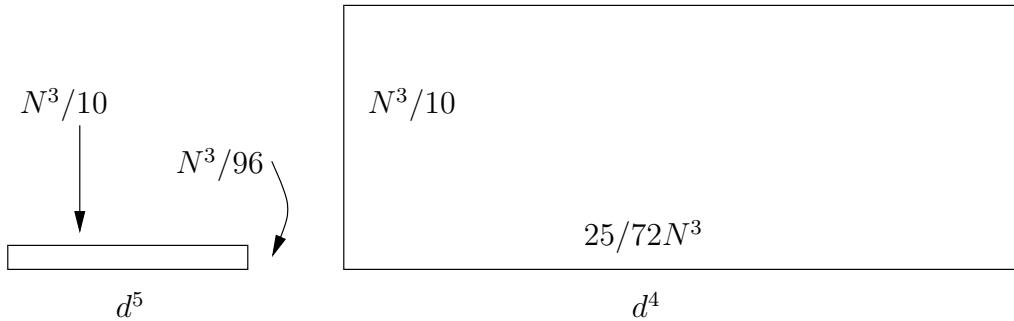


FIGURE 1. Relative sizes of the matrices d^5 , d^4 in our cochain complex. We will exploit the fact that d^4 is wider than its height.

Given an $m \times n$ sparse matrix A , our fundamental computational task is to find the Smith normal form (SNF) decomposition

$$(2) \quad A = PDQ.$$

Here $P \in \mathrm{GL}_m(R)$, $Q \in \mathrm{GL}_n(R)$, D is zero except on the diagonal, and the nonzero diagonal entries satisfy $d_{ii} \mid d_{i+1,i+1}$ for all i . There is a ρ with $d_{ii} \neq 0$ for $0 \leq i < \rho$ and $d_{ii} = 0$ for $i \geq \rho$; when R is a field, ρ is the rank of A . We call P and Q *change of basis* matrices.

To carry out the calculations, we used SHEAFHOM 2.2, a free software package written by the third author [McCb, McCa]. SHEAFHOM performs large-scale computations in the category of finitely-generated R -modules, where R is any principal ideal domain supporting exact computation. Most development work in SHEAFHOM has been for domains that are not fields, especially \mathbb{Z} and other rings of integers of class number 1. In this sense it differs from most of the sparse matrix literature, which looks at \mathbb{R} and \mathbb{C} [DER89, GVL96, Dav06, Mat03] or finite fields [Wie86, LO91, PS92, Tei98]. The differences are because we need to compute P and Q , as explained in the introduction. For matrices over \mathbb{Z} , one can find the SNF D matrix efficiently by reducing modulo a number of primes [DSV01] [DE-VGU07], or by other techniques [HHR93] [BCP97]. Yet it is not clear how to find P and Q by reducing by multiple primes. The need for the change of basis matrices is why SHEAFHOM aims to work globally.

Fill-in is a concern in sparse linear algebra over any ring R . Imagine two vectors that both have a nonzero at i_0 . Add a scalar multiple of the first to the second in

order to zero out the value at i_0 . In general, the result will have nonzeros in the union of the positions where the originals had nonzeros, apart from i_0 . (We follow the standard convention in the sparse matrix literature and abbreviate “nonzero entries” to “nonzeros.”) For very sparse vectors, the number of nonzeros almost doubles. Fill-in is this growth in the number of nonzeros.

A separate issue when R is infinite is *integer explosion*. Over \mathbb{Z} , the length (number of digits) of the product of two numbers is roughly the sum of the lengths of the factors. A vector operation that zeroes out one position will tend to increase the length of the numbers in the rest of the vector. SHEAFHOM’s purpose is to avoid fill-in and integer explosion as much as possible. With $R = \mathbb{F}_p$, integer explosion is not an issue, and the focus is on avoiding fill-in.

We find the SNF by performing a sequence of *elementary operations*: permuting two columns, adding a scalar multiple of one column to another, and multiplying a column by a unit of R , plus the same for rows. The algorithm is described in (2.1.2).

Algorithms that operate on only one side of the matrix are more familiar. These include the Hermite normal form (HNF) $A = HQ$ [Coh93, (2.4.2)]. Over a field, HNF is the same as Gaussian elimination on columns, with H in column-echelon form and $Q \in GL_n(\mathbb{Z})$.

In principle, we prefer SNF to HNF because we are working with cochain complexes. To evaluate Hecke operators on H^i , we need to compute with the map $\ker d^i \rightarrow (\ker d^i)/(\text{im } d^{i-1})$ that reduces cocycles modulo coboundaries. This requires the P matrix of d^{i-1} and the Q matrix of d^i . When computing all of H^* , it is natural to compute P and Q for d^* at the same time.

When the d^i are very large, however, we must compromise by omitting computation of the change-of-basis matrices that are not needed. Since this paper is about H^5 , we compute for d^5 only D and Q , and for d^4 only P and D . The biggest savings arise because the largest matrices, d^4 , are significantly wider than their height, as Figure 1 shows. The Q matrices for d^4 , those on the longer side, are fortunately the ones we can forget.

HNF does have one advantage over SNF when one is forgetting the Q matrix: it can be computed by the following *disk HNF* algorithm. Write the whole matrix A to disk, then read it back in one column at a time. As one reads each column, one puts the matrix accumulated so far into HNF. Over $R = \mathbb{F}_p$, this means using standard Gaussian elimination on columns, with no special pivoting algorithm. Again, the savings arise because d^4 has a short side and a long side. The echelon form never exceeds $n_5 \times n_5$, the square on the short side.

With the wrong matrices, though, disk HNF is a recipe for disaster. It can succeed only if the matrix has low co-rank. The *co-rank* of an $m \times n$ matrix is $\min(m, n)$ minus the rank ρ of the matrix. Assume $m \leq n$ from now on (this holds for our d^5 and d^4), so the co-rank is $m - \rho$. Imagine that, after some work, one must put such a matrix into column-echelon form using Gaussian elimination. We claim that the echelon form will have a great deal of fill-in, no matter how cleverly the pivots are chosen. The echelon form will have ρ pivot rows with only a single nonzero. The remaining $m - \rho$ rows will in general be dense—no pivot has had a chance to clear out their entries, and by the law of averages they will mostly be nonzero. Hence there are about $(m - \rho) \cdot \rho = (\text{co-rank}) \cdot (\text{rank})$ nonzeros in the echelon matrix. We cannot stop ρ from being large. But when $m - \rho$ is also large, the product $(m - \rho) \cdot \rho$ is prohibitively large. These observations are for

the final result of a computation; usually fill-in is even worse in the middle of the computation, before all the pivot rows have been established.

The main technical observation of this paper is to use the change of basis matrices in a simple way to transform d^4 into an equivalent matrix η of low co-rank. We start with SNF on η , switch to disk HNF when the fill-in of SNF forces us to do so, and rely on the low co-rank to make disk HNF succeed. The matrix η is defined in Equation 3 below.

In (2.1) and (2.2) we present these ideas more precisely.

2.1. Computing the SNF. Let A by an $m \times n$ matrix with entries in a field R where exact computation is possible. We define *elementary matrices* $P_l \in \mathrm{GL}_m(R)$ as usual [Jac85, p. 182]. These are permutation, translation, and dilation matrices, corresponding to the elementary operations listed above. Replacing A with $P_l A$ or $P_l^{-1} A$ performs an elementary row operation on A . Multiplying on the right by an elementary matrix $Q_l \in \mathrm{GL}_n(R)$ performs an elementary column operation.

2.1.1. The Markowitz algorithm. Markowitz [DER89, (7.2)] [HHR93] is a well-established approach to reducing fill-in. It is a greedy algorithm, reducing fill-in as much as possible at each step. Let a_{ij} be a nonzero. Let r_i be the number of nonzeros in a_{ij} 's row, and c_j the number in its column. If one adds a multiple of row i to row k in order to zero out the a_{kj} entry, one creates up to $r_i - 1$ new nonzeros in row k . Using row i to clear out the entire column j produces up to $(r_i - 1)(c_j - 1)$ new entries. The Markowitz algorithm, in its simplest form, chooses at each step the pivot a_{ij} that minimizes the *Markowitz count* $(r_i - 1)(c_j - 1)$. (If R were \mathbb{Z} , we would also need to address integer explosion, by avoiding pivots with large absolute value even if they have low Markowitz count.)

It can be slow to compute the Markowitz count for all a_{ij} . One approach is to look at only a few rows with small r_i —say the smallest ten rows—and minimize the Markowitz count only for those rows. Early versions of the SHEAFHOM code used this approach. Currently, we prefer to avoid fill-in at whatever cost in speed, and we always search over all entries. To speed up the search, we store the r_i and c_j in arrays and update the arrays with every elementary operation.

2.1.2. Statement of the algorithm. We now describe SHEAFHOM 2.2's SNF algorithm. Implementation details are deferred to (2.3).

The main strength of the algorithm is the interplay between the Markowitz count and disk HNF when the co-rank is low. Outside these aspects, it is like many SNF algorithms [Jac85, (3.7)] [Coh93, (2.4.4)].

The algorithm uses an index c , the *pivot index* or *corner*, which just says the current pivot is at a_{cc} . The *active region* is where $c \leq i < m$ and $c \leq j < n$. Outside the active region, the matrix has nonzeros on the diagonal and nowhere else.

The parameter τ controls when we switch from SNF to disk HNF. It is chosen by the user based on heuristics and experience; see (2.3.6) for details. The part of the algorithm with τ is stated when we are forgetting the Q 's, as we always do for d^4 ; it would be easy to extend this part for the P 's also.

If one does not need to remember P and Q , one simply omits the part of the algorithm that writes them out. Our implementation overwrites A with D .

Input: an $m \times n$ sparse matrix $A = (a_{ij})$ with entries in a field R . If we are not finding the Q change of basis matrices, we are given in addition a parameter τ (defaulting to $\tau = \infty$).

Output: A finite list (P_0, P_1, P_2, \dots) of $m \times m$ elementary matrices, a finite list (\dots, Q_2, Q_1, Q_0) of $n \times n$ elementary matrices, and an $m \times n$ diagonal matrix D , such that

$$A = (P_0 P_1 P_2 \cdots) \cdot D \cdot (\cdots Q_2 Q_1 Q_0) = PDQ$$

as in Equation 2.

Step 0. Set $c = 0$.

Step 1. If the number of nonzeros in the active region is $\geq \tau$, and if disk HNF has not run yet, run disk HNF on the active region. That is, write the active region to disk, find its column-echelon form while reading it back one column at a time, and replace the old active region in memory with the echelon form, keeping track of the Q 's.

Step 2. If the active region is entirely zero, including the case $c = \min(m, n)$, then return the lists of P 's and Q 's, return A overwritten with D , and terminate the algorithm.

Step 3. Choose a nonzero in the active region that minimizes the Markowitz count. This is the *pivot*. Use a row and column permutation if necessary to move the pivot to the a_{cc} position (this is *complete pivoting*). If the row permutation is $A \rightarrow P_l^{-1}A$, then append P_l to the right side of the list of P 's. (Of course $P_l = P_l^{-1}$ for order-two permutations.) Similarly, append the column permutation Q_l to the left side of the list of Q 's.

Step 4. For all j with $c < j < n$ and $a_{cj} \neq 0$, subtract a multiple of column c from column j to make $a_{cj} = 0$. For each of these elementary operations $A \rightarrow A Q_l^{-1}$, append Q_l to the left side of the list of Q 's.

Step 5. For all i with $c < i < m$ and $a_{ic} \neq 0$, subtract a multiple of row c from row i to make $a_{ic} = 0$. For each of these elementary operations $A \rightarrow P_l^{-1}A$, append P_l to the right side of the list of P 's. (If R were not a field, steps 4 and 5 would need to be extended when a_{ij}/a_{cc} has a nonzero remainder for some a_{ij} in the active region.)

Step 6. Increment c and go to step 1.

2.1.3. Representing change-of-basis matrices. It is important that we return P and Q as lists of elementary matrices. The products $P_0 P_1 P_2 \cdots$ and $\cdots Q_2 Q_1 Q_0$ are likely to be dense; we could never hold them in RAM, much less compute their inverses. Fortunately, it is easy to work with them as lists. Given a matrix B , compute QB by multiplying B on the left by Q_0, Q_1 , and so on. To compute $Q^{-1}B$, run through the Q_l in the opposite order, decreasing l , and multiply B on the left by Q_l^{-1} . Similar comments apply to the P_l , and to transposes.

The lists are still too big to fit in RAM, so we store them on disk. We read once through them every time we need to operate with P or Q . We use a text-based data format where each elementary matrix takes up only about 20 characters. Storing a translation matrix, for example, only requires storing the two indices and the value of the off-diagonal entry. Reading the elementary matrices in left-to-right order is straightforward. To read in right-to-left order, we use a pointer that steps backward through the file.

2.1.4. *Sizes of the matrices.* We will need more precise estimates for the $n_i = \dim C^i$. We refer the reader to [AGM02] for the definitions. The n_i are approximated by sums of terms of the form $|\mathbf{P}^3(\mathbb{Z}/N\mathbb{Z})|/|\Gamma_\sigma|$. Here $\mathbf{P}^3 = \mathbf{P}^3(\mathbb{Z}/N\mathbb{Z})$ is projective three-space over the ring $\mathbb{Z}/N\mathbb{Z}$, and Γ_σ are the stabilizer groups of certain basis cells in the well-rounded retract W .

$$\begin{aligned} n_6 &\approx \frac{1}{96} |\mathbf{P}^3| \\ n_5 &\approx \left(\frac{1}{24} + \frac{1}{24} + \frac{1}{60} \right) |\mathbf{P}^3| = \frac{1}{10} |\mathbf{P}^3| \\ n_4 &\approx \left(\frac{1}{8} + \frac{1}{6} + \frac{1}{24} + \frac{1}{72} \right) |\mathbf{P}^3| = \frac{25}{72} |\mathbf{P}^3| \end{aligned}$$

If N has the prime factorization $\prod p_i^{e_i}$, then

$$|\mathbf{P}^3| = N^3 \prod \frac{1 + p_i + p_i^2 + p_i^3}{p_i^3}.$$

When N is prime, this reduces to the familiar formula $|\mathbf{P}^3| = 1 + N + N^2 + N^3$. In general, if we consider the set $\{p_i\}$ of primes dividing N to be fixed, then $|\mathbf{P}^3|$ is a constant times N^3 , where the constant depends on the set. In the range of N we consider, up to the low 200's, the largest constant we encounter is 4.04, for $N = 2 \cdot 3 \cdot 5 \cdot 7 = 210$. This factor of 4 is why we said that our estimates for the n_i are a few times larger for composite N than for prime N .

The \approx symbols arise because the orbits of Γ_σ in W are generically of size $|\Gamma_\sigma|$ but are occasionally smaller. What is more, we only count the orientable orbits. Experiments with N in the range 40 to 100 suggest that the error in the \approx is at worst 0.3% for prime N and 0.6% for composite N .

2.2. **Computing cohomology.** Let d^5 and d^4 be as in Formula 1, with $n_i = \dim C^i$. We now describe how we use SHEAFHOM to compute H^5 of the complex.

First, compute the SNF $d^5 = (?)D_5Q_5$ with P_5 omitted. Let $\rho_5 = \text{rank } D_5 = \text{rank } d^5$.

Second, define

$$(3) \quad \eta = Q_5 d^4.$$

Since d^* is a cochain complex, the topmost ρ_5 rows of η are zero. Delete these rows, still calling the matrix η . Thus η has dimension $(n_5 - \rho_5) \times n_4$.

Third, compute the SNF $\eta = P_\eta D_\eta (?)$, with Q_η omitted. Let $\rho_\eta = \text{rank } D_\eta = \text{rank } \eta$. Note that $\text{rank } d^4 = \rho_\eta$, since d^4 and η differ only by a change of basis and deleting some zero rows.

We can now report the Betti number $h_5 = \dim H^5$:

$$h_5 = n_5 - \rho_5 - \rho_\eta.$$

We need not only the Betti number, though, but an explicit list z_1, \dots, z_{h_5} of cocycles in $\ker d^5$ that descend modulo $\text{im } d^4$ to a basis of the cohomology. Let B be the $(n_5 - \rho_5) \times h_5$ matrix with the identity in the bottom $h_5 \times h_5$ block and zeroes elsewhere. Compute $\bar{B} = P_\eta B$. Add to the top of \bar{B} the ρ_5 rows of zeroes that we deleted from η , still calling it \bar{B} . Then the columns of $Q_5^{-1} \bar{B}$ are an appropriate list of cocycles z_j .

Our Hecke operator algorithm takes a cocycle y and maps it to its Hecke translate, a cocycle y' . For simplicity, assume $y = z_j$. The Hecke translate z'_j is generally not in the span of the z_j . Rather, $z'_j = s_{1,j}z_1 + \dots + s_{h_5,j}z_{h_5} + b_j$, where the $s_{i,j} \in R$ are scalars and $b_j \in \text{im } d^4$ is a coboundary. Computing the $s_{i,j}$ and b_j is straightforward, since Q_5 , P_η , and the z_j are all stored on disk. Ultimately, we express each Hecke operator as the $h_5 \times h_5$ matrix $(s_{i,j})$ with respect to our cohomology basis.

2.2.1. Co-rank of η . Using η may seem inelegant because it violates the symmetry of a cochain complex. Since the complex is

$$0 \longleftarrow C^6 \xleftarrow{P_5 D_5 Q_5} C^5 \xleftarrow{P_4 D_4 Q_4} C^4 \longleftarrow \dots,$$

it is more elegant to compute H^5 using Q_5 and P_4 , which are both $n_5 \times n_5$ matrices.

However, η has one great virtue: by removing the rows of zeroes from its top, we have dropped its co-rank down to h_5 . We observe that h_5 is never more than 80 for the N we consider, while ρ_5 could reach into the millions. This difference is what allows disk HNF to succeed.

Let us give more precise estimates. The Betti number $h_6 = \dim H^6$ equals $n_6 - \rho_5$ since our chain complex has only 0's after degree 6. We observe that h_6 is never more than about 40 in our range of N . Thus $\rho_5 \approx N^3/96 - 40$. Estimating h_5 as 80, the rank $\rho_\eta = \rho_4 \approx N^3/10 - (N^3/96 - 40) - 80$. Both 40 and 80 are negligible for large N , so $\rho_\eta \approx (1/10 - 1/96)N^3 = 43/480N^3$. For η , the co-rank is again about 80, meaning the number of entries in η 's echelon form is $(\text{co-rank}) \cdot (\text{rank}) \approx 80 \cdot \rho_\eta = 80 \cdot (43/480)N^3 \approx 7N^3$. But the number of entries in d^4 's echelon form is $\approx (n_5 - \rho_4)\rho_4 \approx ((1/10)N^3 - (43/480)N^3) \cdot ((43/480)N^3) = (1/96)(43/480)N^6 \approx 0.0009N^6$. At $N \approx 200$, the latter is $\approx 7500N^3$. In other words, the echelon form of d^4 has about 1000 times more entries than the echelon form of η when N is near 200.

This analysis was for Gaussian elimination, not SNF. When we compute the SNF of matrices of large co-rank, we observe the same behavior empirically. Figure 2 compares the fill-in for the SNF computations of d^4 and η at the same level $N = 53$. Both matrices have 52766 columns and rank 13614. The example uses Markowitz only, not disk HNF. We show only the pivot indices $c \geq 12000$, since the graphs look the same to the left of that point. The fill-in for d^4 is clearly the worse of the two, with a peak over three times higher than for η . In general, the SNF algorithm displays “catastrophic cancellation”: the number of nonzeros in the active region tends to grow rapidly until almost the end of the algorithm, when the number decreases sharply to zero. Catastrophic cancellation begins for d^4 at row 13464 and for η at about row 13084.

The fill-in for our smaller matrix d^5 is harder to predict. There are many columns with only one or two entries. These allow Markowitz to reduce the matrix with no fill-in at all, at least in the initial stages. Later, the number of nonzeros grows rapidly as for d^4 , with an even more precipitous cancellation at the end. Figure 3 gives the example for level $N = 103$.

2.3. Implementation. SHEAFHOM 2.2 is written in Common Lisp, which is the ideal language for it in many ways. Common Lisp source code compiles to very efficient machine code; carefully-written Lisp is as fast as C++ [Gat00]. Yet writing high-level code in Lisp is easier than in C or C++. Like its inheritor Python,

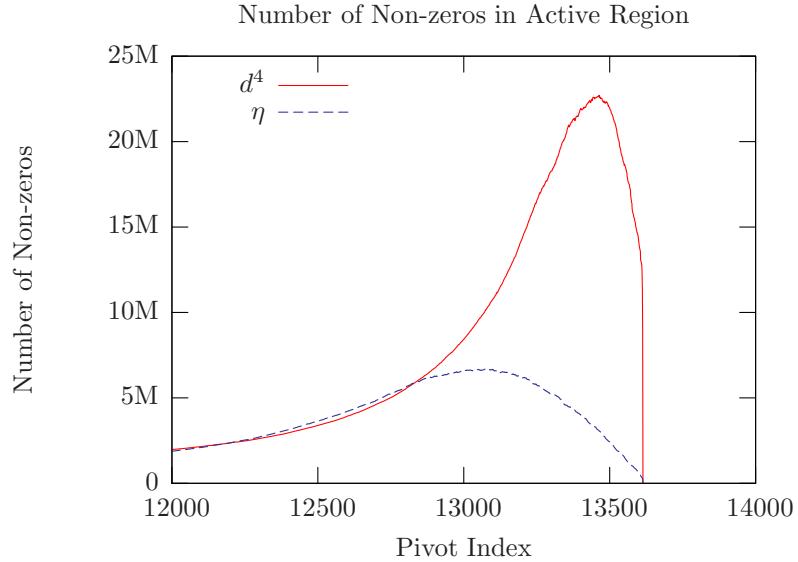


FIGURE 2. Size of the active region during an SNF computation for d^4 and η . Here M denotes million.

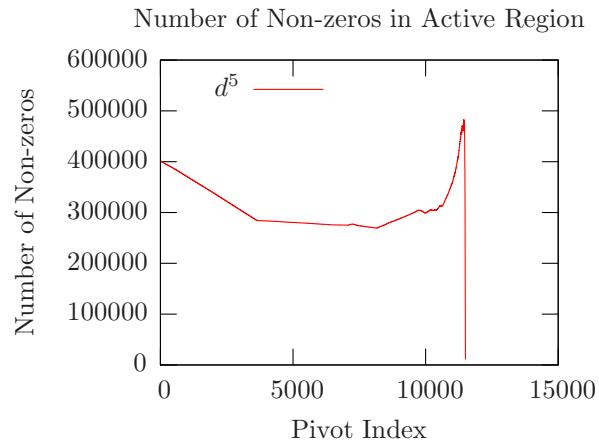


FIGURE 3. Size of the active region during an SNF computation for d^5 .

Lisp has an outer real-eval-print loop that promotes high-level thinking and rapid development. Type safety is built in. Lisp's object-oriented package CLOS is the most flexible of any language's. Arbitrary-precision integers and rationals are part of the language standard and can be very fast.

SHEAFHOM 2.1 and 2.2 were developed with Allegro Common Lisp (ACL) from Franz, Inc. Under Linux we use the free implementation Steel Bank Common Lisp (SBCL). SHEAFHOM 2.0 was developed in Java. SHEAFHOM 1.x was developed in

Carnegie-Mellon University Common Lisp (CMUCL), the parent of SBCL. All the Lisps mentioned here produce high-quality compiled code.

SHEAFHOM 1.x was restricted to $R = \mathbb{Q}$, but had general support for the derived category of complexes of sheaves over cell complexes. It was also ported to Macaulay 2 [GS].

2.3.1. Overview of data structures. Any sparse matrix package uses data structures that store only the nonzeros. SHEAFHOM stores a sparse matrix as an array of sparse vectors representing the columns. A sparse vector is a linked list of nonzero sparse elements. A *sparse element* represents an ordered pair (i, v) with $i \in \mathbb{N}$, $v \in R$. When this sparse element occurs in column j , it means the i, j entry of the matrix has nonzero value v . The rest of (2.3) gives further details.

2.3.2. Testing platforms. To explain the test data we will present later, we give some details about the two machines we use most. Portland, our Windows machine, is a laptop with a 1.30 GHz Celeron M processor and 992 MB of RAM, running ACL 8.1 under Windows XP 2002. Athens, our Linux machine, is a 64-bit laptop with a 2.20 GHz Intel Core2 Duo processor and 3.9 GB of RAM, running SBCL 1.0.12 under Linux 2.6 (Ubuntu 8).

2.3.3. Implementation of sparse elements. For general rings R , SHEAFHOM implements the indices $i \in \mathbb{N}$ and values $v \in R$ as objects. They are allocated on the heap, and contain type information needed to use them in an object-oriented way. For these rings, we store (i, v) as a pair of pointers to i and to v . The pair of pointers is a native data structure in Lisp, the *cons*, the fundamental building block for linked lists and tree structures.

The cons implementation is convenient, but, as in all languages, the indirection carries a penalty in efficiency. Lisp may store the numbers on a different memory page from the conses themselves, forcing the machine to flip back and forth between pages.

When $R = \mathbb{F}_p$, we implement sparse elements more efficiently. We typically use primes p a little less than 2^{15} , so that sums and products in $[0, p)$ can be computed in a 32-bit integer before reducing mod p . Let k be the smallest integer such that $p < 2^k$. (For our favorite prime 12379, $k = 14$.) Each $v \in \mathbb{F}_p$ fits in k bits. We would like to pack (i, v) into a single 32-bit integer, but we cannot. With $k = 14$, say, there would only be 18 bits left for the index, and we need a larger range of indices than $2^{18} \approx 10^6$. Therefore, on a 32-bit machine, we pack (i, v) into an arbitrary-precision integer, but arrange things so the integer will fit into 32 bits at the critical stages of the computation. Let M be an upper bound on the number of rows in the matrix. We store (i, v) as the integer

$$(4) \quad (M - i - 1) \cdot 2^k + v.$$

Near the end of a computation, when space is critical, i will be close to M . Hence $M - i - 1$ will be small and Formula 4 will fit in 32 bits.

On a 64-bit machine, we store (i, v) even more simply as

$$(5) \quad i \cdot 2^k + v.$$

For us, this never exceeds a 64-bit integer. When $k = 14$, for instance, it would exceed it only if $i > 2^{64-14} = 2^{50} \approx 10^{15}$, whereas our largest m are around 10^6 or 10^7 . By declaring the sparse element to be a 64-bit integer throughout the

program, we avoid the separate heap allocations for i and v , the pair of pointers holding i and v , and the object-oriented indirection.

In all of SHEAFHOM's implementations of sparse elements, operations in the ring R really operate on the sparse elements. For instance, the sum of (i_1, v_1) and (i_2, v_2) is a new sparse element with value $v_1 + v_2 \in R$ and with some i determined by the context, typically $i = i_1$.

2.3.4. Implementation of sparse vectors and matrices. SHEAFHOM's sparse matrix data structure is an array of sparse vectors giving the columns, together with arrays of the row lengths r_i and column lengths c_j used for Markowitz. SHEAFHOM implements a sparse vector as a singly linked list of sparse elements, sorted by increasing index i . The backbone of the linked list (as opposed to the sparse elements it holds) is again built from conses. Implementing vectors as a linked list is flexible, compared to constantly allocating and reallocating arrays in C. Lists grow in size as fill-in occurs, and shrink as entries cancel each other out. As they shrink, the memory is reclaimed by the garbage collector.

SHEAFHOM includes a macro `with-splicer` as a mini-language for surgery on lists. `with-splicer` iterates down a list and offers read, insert, modify and delete operations for individual elements, plus splicing commands to add and remove larger chunks. We allocate as few new backbone conses as possible.

Implementing sparse vectors as singly-linked lists is flexible, as we have said, but it involves the usual risks because accessing a single element might take time linear in the length of the list. We can avoid this trap with a little care. To find the sum $\mathbf{x} + \mathbf{y}$ of sparse vectors \mathbf{x} and \mathbf{y} , for example, we run through the vectors in parallel and accumulate the sum as we go. Similar comments apply to scalar multiplication, subtraction, and the dot product.

One place we might encounter quadratic behavior is step 5 of the SNF algorithm (2.1.2). If the pivot row and column have r_i and c_j entries respectively, a direct approach would require $(r_i - 1)(c_j - 1)$ operations, each linear in the length of the column. The trick here is simply to put step 4 before step 5. Step 4 handles the column operations linearly as in the previous paragraph, and step 5 then has $r_i = 1$.

Another likely place for slow behavior is multiplying a sparse matrix B on the left by an elementary matrix. This includes the important computation of η (Equation 3). An elementary operation with rows i and j might involve a linear sweep down $r_i + r_j$ columns. We handle this situation by taking the transpose of B , multiplying on the right by the transposes of the elementary matrices, and taking a final transpose to put everything back at the end. The transpose of B takes time linear in the number of entries, which for a sparse matrix is small by definition.

2.3.5. Comparison of sparse element implementations. Table 1 shows a test of speed and space requirements for the three implementations of sparse elements over \mathbb{F}_{12379} on our Linux machine. We timed the SNF computation for the d^4 matrix for level $N = 53$, the matrix of Figure 2. The matrix is 15218×52766 . We used only the Markowitz portion of (2.1.2), no disk HNF. Since we were on a 64-bit machine, each sparse element in Formula (5) takes 8 bytes. Formula (4) has about the same space requirement, especially towards the end of a computation when i is close to M . The (i, v) representation requires 8 bytes for each of i and v , plus 16 bytes for the cons itself, for a total of 32 bytes. To all these figures we must add

16 bytes for the cons in the backbone of the sparse vector that holds the sparse element.

2.3.6. Overall speed and space requirements. To summarize, our implementations of sparse elements are optimized for both space and speed, and our sparse vector and matrix algorithms avoid speed traps leading to quadratic time unnecessarily. On the other hand, at the higher layers of the algorithm, we sacrifice speed, minimizing fill-in at all costs. For instance, we mentioned in (2.1.1) that we do a full Markowitz scan at each pivot step. This takes about one third of the total time for the algorithm until we switch to disk HNF.

The largest matrix we have handled so far has size $845,712 \times 3,277,686$. This is the η matrix for level $N = 211$. It has close to 20 million nonzeros. We carried out the computation on our Linux machine, using the implementation of \mathbb{F}_{12379} in Formula 5. The sizes of d^5 and d^4 are $98,351 \times 944,046$ and $944,046 \times 3,277,686$, respectively. We reduced d^5 using only the Markowitz portion of (2.1.2), with no disk HNF. We reduced η using both Markowitz and disk HNF, switching when the active region had τ equal to about 116 million nonzeros. Converting d^4 to η as in (3) took significant time by itself, since it called for over three million elementary column operations on the transpose of d^4 . How the running time broke down is shown in Table 2.

It is interesting to compare our running times with those in [DEVGU07]. They compute for $GL_7(\mathbb{Z})$ at level $N = 1$, while we do $SL_4(\mathbb{Z})$ at $N = 211$. The number of nonzeros is roughly comparable, 37 million versus 20 million. Both computations took about one month. They computed mod p for several p , but used 50 processors; we did one p on one processor. We find ourselves joking that

$$GL_7 = GL_4 + 211^3.$$

How the running time broke down is shown in Table 2. We do not distinguish between the wall-clock time and CPU time because they are essentially the same. We ran the computation on our Linux machine on a single processor. The machine was doing nothing besides the computation. Occasional tests with `top` showed the CPU running consistently at 100%. We presume one of the two cores ran the computation, and the other took care of background jobs like displaying the desktop.

Recall that τ is the maximum number of nonzeros allowed in the active region before switching from Markowitz to disk HNF. Table 3 shows the largest τ we have used successfully. They depend on the chosen implementation of sparse elements, as well as on the operating system and version of Lisp. A + means we have relatively little data for this combination of parameters, and τ could likely go higher than shown. Values without a + represent a reasonable maximum, determined by long experience and many out-of-memory crashes. The number of bytes is computed as in (2.3.5) for our Linux machine. On our Windows machine, a 32-bit integer counts 4 bytes, a cons 8 bytes.

2.3.7. Other approaches. We mention a few more sparse matrix techniques that appear in the literature.

Many scientific applications involve sparse matrices with a pattern, such as tridiagonal or banded diagonal. The matrices in this paper have no recognizable sparsity

implementation	total time	space per entry
(i, v) as a cons	2768 sec	48 bytes
Formula 4	1385 sec	24 bytes
Formula 5	784 sec	24 bytes

TABLE 1. Comparison of sparse element implementations.

SNF of d^5	$\frac{1}{4}$ day
converting d^4 to η	$5\frac{1}{2}$ days
SNF of η , Markowitz portion	12 days
SNF of η , disk HNF portion	13 days

TABLE 2. Overall speed for level $N = 211$.

machine and RAM	implementation	largest τ	space per entry	total space
Windows 1GB	(i, v) as a cons	30M	24	720MB
Windows 1GB	(4)	22M+	12	264MB+
Linux 4GB	(4)	42M+	24	1.0GB+
Linux 4GB	(5)	148M	24	3.55GB

TABLE 3. Maximum number of nonzeros allowed in the active region.

pattern. A matrix coming from a d -dimensional topological space would have a pattern in d dimensions, but not when flattened into a two-dimensional table of rows and columns.

The RSA challenge matrices described in [PS92] had some special properties. The columns on the left were very sparse, and could be used to clean out the somewhat denser columns on the right. Rows and columns with only two entries gave an even quicker reduction step [LO91]. The [DEVGU07] matrices had many rows with only one entry [DEVGU07, 2.6.4], a result of cancellation at the low level $N = 1$. By and large, our matrices do not have these properties. The sparsity is almost entirely uniform. The d^5 have a substantial fraction of columns with one or two entries, but not d^4 .

Block-triangularization is another well-established technique for sparse matrices. Given an $m \times n$ matrix A , we look for permutation matrices P_b and Q_b so that $B = P_b A Q_b$ is *block upper-triangular*: it has square blocks down the diagonal and nothing but zeroes below the blocks. The matrix can then be reduced one block at a time, either to HNF or SNF. The upper-triangular part can be handled directly by back-solving. Since we only permute the matrix entries, there is no fill-in and no integer explosion. Assume for the moment that A is square and of full rank, and that after a row permutation the diagonal a_{ii} is all nonzero. For such A , the block-triangular form is unique, and the way to find it is well known [DER89, Ch. 6]. When A is not of full rank, the block decomposition is generalized to the Dulmage-Mendelsohn decomposition, which is roughly upper-triangular [Dav06, (8.4)]. In our case, A is a fraction of a percent away from full rank and from having nonzeros down the diagonal; for square A of this type,

finding the Dulmage-Mendelsohn decomposition takes about the same time and space resources as block decomposition. So far these algorithms are polynomial time. A new idea is needed, however, when A is not square but rectangular, as it is for us. One can find the Dulmage-Mendelsohn decomposition of the left-hand $m \times m$ square, then swap in columns from the wide section on the right in a way that shrinks the left-hand diagonal blocks. Deciding which columns to swap is in general an NP-hard problem. The third author has found good decompositions of some rectangular A using a heuristic for which columns to swap in. One iterates the procedure “do Dulmage-Mendelsohn, then swap in columns” many times. We defer the details to a future paper.

3. EISENSTEIN COHOMOLOGY AND PARAMODULAR FORMS

In this section we provide the necessary background to state Conjectures 1 and 2 and explain our computational results in Section 4.

3.1. Hecke eigenclasses and Galois representations. We will describe some classes appearing in $H^5(\Gamma_0(N); \mathbb{C})$ in terms of the Galois representations conjecturally attached to them. Thus we begin by recalling what this means [AGM08].

Let $\xi \in H^5(\Gamma_0(N); \mathbb{C})$ be a Hecke eigenclass. In other words, ξ is an eigenvector for certain operators

$$T(l, k): H^5(\Gamma_0(N); \mathbb{C}) \rightarrow H^5(\Gamma_0(N); \mathbb{C}),$$

where $k = 1, 2, 3$ and l is a prime not dividing N . These operators correspond to the double cosets $\Gamma_0(N)D(l, k)\Gamma_0(N)$, where $D(l, k)$ is the diagonal matrix with $4 - k$ ones followed by k l 's. (One can also define analogues of the operators U_l for $l \mid N$, although we do not consider them in this paper.) Suppose the eigenvalue of $T(l, k)$ on ξ is $a(l, k)$. We define the Hecke polynomial $H(\xi)$ of ξ by

$$(6) \quad H(\xi) = \sum_k (-1)^k l^{k(k-1)/2} a(l, k) T^k \in \mathbb{C}[T].$$

Now we consider the Galois side. Let $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of \mathbb{Q} . Let $\rho: \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_n(\bar{\mathbb{Q}}_p)$ be a continuous semisimple Galois representation unramified outside pN . Fix an isomorphism $\varphi: \mathbb{C} \rightarrow \bar{\mathbb{Q}}_p$. Then we say the eigenclass ξ is *attached* to ρ if for all l not dividing pN we have

$$\varphi(H(\xi)) = \det(1 - \rho(\text{Frob}_l)T),$$

where $\text{Frob}_l \subset \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is the Frobenius conjugacy class over l . Let ε be the p -adic cyclotomic character, so that $\varepsilon(\text{Frob}_l) = l$ for any prime l coprime to p . We denote the trivial representation by i .

3.2. Eisenstein cohomology. Let $X = \text{SL}_4(\mathbb{R})/\text{SO}(4)$ be the global symmetric space, and let X^{BS} be the partial compactification constructed by Borel and Serre [BS73]. The quotient $Y := \Gamma_0(N) \backslash X$ is an orbifold, and the quotient $Y^{\text{BS}} := \Gamma_0(N) \backslash X^{\text{BS}}$ is a compact orbifold with corners. We have

$$H^*(\Gamma_0(N); \mathbb{C}) \simeq H^*(Y; \mathbb{C}) \simeq H^*(Y^{\text{BS}}; \mathbb{C}).$$

Let $\partial Y^{\text{BS}} = Y^{\text{BS}} \setminus Y$. The Hecke operators act on the cohomology of the boundary $H^*(\partial Y^{\text{BS}}; \mathbb{C})$, and the inclusion of the boundary $\iota: \partial Y^{\text{BS}} \rightarrow Y^{\text{BS}}$ induces a map on cohomology $\iota^*: H^*(Y^{\text{BS}}; \mathbb{C}) \rightarrow H^*(\partial Y^{\text{BS}}; \mathbb{C})$ compatible with the Hecke action.

The kernel $H_!^*(Y^{\text{BS}}; \mathbb{C})$ of ι^* is called the *interior cohomology*; it equals the image of the cohomology with compact supports. The goal of Eisenstein cohomology is to use Eisenstein series and cohomology classes on the boundary to construct a Hecke-equivariant section $s: H^*(\partial Y^{\text{BS}}; \mathbb{C}) \rightarrow H^*(Y^{\text{BS}}; \mathbb{C})$ mapping onto a complement $H_{\text{Eis}}^*(Y^{\text{BS}}; \mathbb{C})$ of the interior cohomology in the full cohomology. We call classes in the image of s *Eisenstein classes*. (In general, residues of Eisenstein series can give interior, noncuspidal cohomology classes, with infinity type a Speh representation, but as noted in [AGM02], these do not contribute to degree 5.)

3.3. Paramodular forms. We now give some background on Siegel modular forms. We will skip the basic definitions, which can be found in many places (e.g. [vdG08]), and instead emphasize paramodular forms. Since our main application is cohomology of subgroups of $\text{SL}_4(\mathbb{Z})$, we will focus on Sp_4 .

Let $K(N)$ be the subgroup of $\text{Sp}_4(\mathbb{Q})$ consisting of all matrices of the form

$$\begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

The group $K(N)$ is called the *paramodular group*. It contains as a subgroup the standard congruence subgroup $\Gamma'_0(N)$ modeled on the Klingen parabolic; that is, $\Gamma'_0(N) \subset \text{Sp}_4(\mathbb{Z})$ is the intersection $K(N) \cap \text{Sp}_4(\mathbb{Z})$. *Paramodular forms* are Siegel modular forms that are modular with respect to $K(N)$. Clearly such forms are also modular with respect to $\Gamma'_0(N)$, although modular forms on the latter are not necessarily paramodular. Note also that in the embedding $i: \text{Sp}_4(\mathbb{Z}) \rightarrow \text{SL}_4(\mathbb{Z})$, we have $i(\Gamma'_0(N)) = i(\text{Sp}_4(\mathbb{Z})) \cap \text{SL}_4(\mathbb{Z})$.

The paramodular forms of interest to us are those of prime level N and weight 3. We denote the complex vector space of such forms by $P_3(N)$. One can show that $P_3(N)$ consists entirely of cuspforms, i.e. there are no weight 3 paramodular Eisenstein series. Recently T. Ibukiyama [Ibu07] proved a general dimension formula for $P_3(N)$:

Theorem 1. [Ibu07, Theorem 2.1] *Let N be prime and let $\kappa(a)$ be the Kronecker symbol $(\frac{a}{N})$. Define functions $f, g: \mathbb{Z} \rightarrow \mathbb{Q}$ by*

$$f(N) = \begin{cases} 2/5 & \text{if } N \equiv 2, 3 \pmod{5}, \\ 1/5 & \text{if } N = 5, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(N) = \begin{cases} 1/6 & \text{if } N \equiv 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\dim P_3(2) = \dim P_3(3) = 0$. For $N \geq 5$, we have

$$\begin{aligned} \dim P_3(N) &= (N^2 - 1)/2880 \\ &+ (N + 1)(1 - \kappa(-1))/64 + 5(N - 1)(1 + \kappa(-1))/192 \\ &+ (N + 1)(1 - \kappa(-3))/72 + (N - 1)(1 + \kappa(-3))/36 \\ &+ (1 - \kappa(2))/8 + f(N) + g(N) - 1. \end{aligned}$$

For any N , the space of weight k paramodular forms contains a distinguished subspace $P_3^G(N)$ originally constructed by Gritsenko [Gri95]. This space is defined by a lift from the space $J_{k,N}^{\mathrm{cusp}}$ of cuspidal *Jacobi forms* of weight k and index N to $P_3(N)$. We will not define Jacobi forms here, and instead refer the reader to [EZ80] for background. For our purposes, we will only need to know the dimension $\dim P_3^G(N) = \dim J_{3,N}^{\mathrm{cusp}}$. Formulas for the dimensions of spaces of Jacobi forms can be found in [EZ80, pp. 121, 131-132]; the following reformulation is due to N. Skoruppa:

Theorem 2. *We have*

$$(7) \quad \dim J_{3,N}^{\mathrm{cusp}} = \sum_{j=1}^{m-1} s(k+2j-1) - \left\lfloor \frac{j^2}{4m} \right\rfloor,$$

where $s(k)$ is the dimension of the space of cuspidal elliptic modular forms of full level and weight k .

Let $P_3^{\mathrm{nG}}(N)$ be the Hecke complement in $P_3(N)$ of the subspace $P_3^G(N)$ of Gritsenko lifts. The dimension of this space is easily determined by Theorems 1 and 2.

We conclude our discussion of paramodular forms by defining the Hecke polynomial attached to an eigenform. More details can be found in [PY09]. Let l be a prime not dividing N . Then associated to l there are two Hecke operators T_l and T_{l^2} . For an eigenform $h \in P_3(N)$ we denote the corresponding eigenvalues by δ_l , δ_{l^2} :

$$T_l h = \delta_l h, \quad T_{l^2} h = \delta_{l^2} h.$$

We define the Hecke polynomial attached to h by

$$(8) \quad H_{\mathrm{Sp}}(h) = 1 - \delta_l T + (\delta_l^2 - \delta_{l^2} - l^2)T^2 - \delta_l l^3 T^3 + l^6 T^4.$$

This polynomial is essentially the local factor at l attached to the spinor L -function for h .

4. CONJECTURES AND COMPUTATIONAL RESULTS

In this section we present two conjectures on the structure of $H^5(\Gamma_0(N); \mathbb{C})$ for N prime, and conclude by describing our computational evidence for them.

4.1. Notation. We begin by fixing notation for the different constituents of the cohomology.

- *Weight two holomorphic modular forms:* Let σ_2 be the Galois representation attached to a holomorphic weight 2 newform f of level N with trivial Nebentypus. Let α be the eigenvalue of the classical Hecke operator T_l on f . Let $\mathrm{IIa}(\sigma_2)$ and $\mathrm{IIb}(\sigma_2)$ be the Galois representations in the first two rows of Table 5 (see p. 21).
- *Weight four holomorphic modular forms:* Let σ_4 be the Galois representation attached to a holomorphic weight 4 newform f of level N with trivial Nebentypus. Let β be the eigenvalue of the classical Hecke operator T_l on f . Let $\mathrm{IV}(\sigma_4)$ be the Galois representation in the third row of Table 5.
- *Cuspidal cohomology classes from subgroups of $\mathrm{SL}_3(\mathbb{Z})$:* Let τ be the Galois representation conjecturally attached to a pair of nonselfdual cuspidal cohomology classes $\eta, \eta' \in H^3(\Gamma_0^*(N); \mathbb{C})$, where $\Gamma_0^*(N) \subset \mathrm{SL}_3(\mathbb{Z})$ is the

congruence subgroup with bottom row congruent to $(0, 0, *)$ modulo N . Let γ be the eigenvalue of the Hecke operator $T_{l,1}$ on η , and let γ' be its complex conjugate. Let $\text{IIIa}(\tau)$ and $\text{IIIb}(\tau)$ be the Galois representations in the last two rows of Table 5.

If f is a weight 2 or weight 4 eigenform as above, or a weight 3 paramodular eigenform, we denote by d_f the degree of the extension of \mathbb{Q} generated by the Hecke eigenvalues of f . We say that two eigenforms f, g are *Galois conjugate* if there is an automorphism $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that the Hecke eigenvalues of f are taken into those of g by σ . We say f, g are *equivalent* if g is a \mathbb{Q} -linear combination of f and its Galois conjugates. We extend these notions in the obvious way to eigenclasses $\eta \in H^3(\Gamma_0^*(N); \mathbb{C})$.

For any modular form of weight k with Fourier expansion $f(z) = \sum_n a_n e^{2\pi i n z}$, let $L(s, f)$ be the Dirichlet series $\sum_n a_n / n^s$. The series $L(s, f)$ can be completed to a function $\Lambda(s, f)$ satisfying a functional equation of the shape $s \rightarrow k - s$.

4.2. Eisenstein cohomology.

Conjecture 1. *Let N be prime. Then the cohomology group $H^5(\Gamma_0(N); \mathbb{C})$ contains the following Eisenstein subspaces:*

- (1) *For each equivalence class of weight two holomorphic newforms of level N , choose a representative f with associated Galois representation σ_2 . Then there are two d_f -dimensional subspaces in the cohomology, one attached to the Galois representation $\text{IIa}(\sigma_2)$, and the other to the Galois representation $\text{IIb}(\sigma_2)$.*
- (2) *For each equivalence class of weight four holomorphic newforms of level N , choose a representative f with associated Galois representation σ_4 . Then if the central special value $\Lambda(2, f)$ vanishes, there is a d_f -dimensional subspace in the cohomology attached to the Galois representation $\text{IV}(\sigma_4)$.*
- (3) *For each equivalence class of nonselfdual cuspidal cohomology classes in $H^3(\Gamma_0^*(p); \mathbb{C})$, $\Gamma_0^*(p) \subset \text{SL}_3(\mathbb{Z})$, choose a representative η and let τ be the conjecturally associated Galois representation. Then there are two d_η -dimensional subspaces, one attached to the Galois representation $\text{IIIa}(\tau)$, and the other to the Galois representation $\text{IIIb}(\tau)$.*

Furthermore, for N prime this is a complete description of the Eisenstein subspace of $H^5(\Gamma_0(N); \mathbb{C})$.

In our earlier paper [AGM08], we also gave a conjecture about some Eisenstein subspaces of H^5 . In fact, for weight 2 modular forms and for SL_3 -cuspidal cohomology, there is no difference between [AGM08, Conjecture 1] and the conjecture here. The new part is in the contribution of the weight 4 modular forms. In [AGM08], our data was only sufficient to suggest that the weight 4 forms f appearing were those whose completed L -functions $\Lambda(s, f)$ have a minus sign in their functional equations. Certainly this contains the subspace of forms whose central special value vanishes, but there are additional forms that also contribute (cf. Example 1 below).

Because of our extensive computations, we feel confident that Conjecture 1 completely describes the Eisenstein subspace for prime level. However, Conjecture 1 is not true for composite N , as already remarked in the paragraph after [AGM08, Example 1].

4.3. Paramodular forms.

Conjecture 2. *For N prime, choose an equivalence class of eigenforms in $P_3^{nG}(N)$, and let h be a representative. Let d_h be the degree of the extension of \mathbb{Q} generated by the eigenvalues of h . Then the cuspidal cohomology $H_{cusp}^5(\Gamma_0(N); \mathbb{C})$ contains a $2d_h$ -dimensional subspace spanned by Hecke eigenclasses. If ξ is an eigenclass in this space, then up to Galois conjugacy the Hecke polynomial $H(\xi)$ of ξ from (6) agrees with the Hecke polynomial $H_{\mathrm{Sp}}(h)$ of h from (8).*

We remark that this equality means that ξ is the functorial lift of h with respect to the natural inclusion of L -groups: ${}^L\mathrm{GSp}_4 \rightarrow {}^L\mathrm{GL}_4$.

4.4. Computational results. These are listed in Table 4, which shows our computed Betti numbers and the dimensions of the constituents of the cohomology predicted by Conjectures 1 and 2. For levels ≤ 101 , we checked that the Hecke polynomial for $l = 2$ is correct.

Example 1. We consider the case $N = 127$. There are two weight 2 eigenforms, with Hecke eigenvalues defining respectively a cubic and a septic field. There are three weight 4 eigenforms, with Hecke eigenvalues defining fields of degrees 1, 13, and 17. The degree 13 eigenform has minus sign in the functional equation of its L -function, which means its central special value vanishes. However, there is also another vanishing at this level: the rational eigenform also has vanishing central special value, vanishing that is not forced by the sign of the functional equation. Thus together these modular forms account for a $2 \times 10 + 14 = 34$ dimensional subspace of $H^5(\Gamma_0(N); \mathbb{C})$.

For the rest of the cohomology, we must consider SL_3 and paramodular contributions. There is no cuspidal cohomology for $\Gamma_0^*(127) \subset \mathrm{SL}_3(\mathbb{Z})$. The space of non-Gritsenko lifts has dimension 3. Thus we see an additional 6-dimensional subspace of H^5 coming from these Siegel modular forms, which means $\dim H^5(\Gamma_0(127); \mathbb{C}) \geq 40$. Indeed our computations indicate that this Betti number equals 40.

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Level	$S_2(N)$	$S_4(N)_0$	SL_3	$P_3^{\text{ng}}(N)$	$H^5(\Gamma_0(N); \mathbb{C})$
83	7	7	0	0	21
89	7	8	2	1	28
97	7	11	0	2	29
101	8	9	0	2	29
103	8	10	0	2	30
107	9	10	0	0	28
109	8	12	0	3	34
113	9	12	0	1	32
127	10	14	0	3	40
131	11	11	0	2	37
137	11	15	0	2	41
139	11	14	0	4	44
149	12	15	0	4	47
151	12	15	0	5	49
157	12	18	0	7	56
163	13	19	0	4	53
167	14	15	0	4	51
173	14	18	0	6	58
179	15	17	0	4	55
181	14	20	0	10	68
191	16	17	0	6	61
193	15	23	0	10	73
197	16	22	0	7	68
199	16	20	0	10	72
211	17	23	0	10	77

TABLE 4. Betti numbers and constituents of Conjectures 1 and 2. The entries are the dimensions of the spaces in the headings, which are as follows: (i) $S_2(N)$ denotes weight 2 cuspidal modular forms of level N and trivial character, (ii) $S_4(N)_0$ denotes weight 4 modular forms of level N , trivial character, and with vanishing central special value, (iii) SL_3 denotes the cuspidal cohomology of the congruence subgroup $\Gamma_0^*(N) \subset \text{SL}_3(\mathbb{Z})$, and (iv) $P_3^{\text{ng}}(N)$ denotes weight 3 paramodular forms of level N that are not Gritsenko lifts. In all cases $2 \times (\text{second} + \text{fourth} + \text{fifth}) + \text{third}$ equals the dimension of $H^5(\Gamma_0(N); \mathbb{C})$.

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IIa	$\sigma_2 \oplus \varepsilon^2 \oplus \varepsilon^3$	$(1 - l^2T)(1 - l^3T)(1 - \alpha T + lT^2)$
IIb	$i \oplus \varepsilon^2 \sigma_2 \oplus \varepsilon$	$(1 - T)(1 - lT)(1 - l^2\alpha T + l^5T^2)$
IV	$\sigma_4 \oplus \varepsilon \oplus \varepsilon^2$	$(1 - lT)(1 - l^2T)(1 - \beta T + l^3T^2)$
IIIa	$\tau \oplus \varepsilon^3$	$(1 - l^3T)(1 - \gamma T + l\gamma' T^2 - l^3T^3)$
IIIb	$i \oplus \varepsilon\tau$	$(1 - T)(1 - l\gamma T + l^3\gamma' T^2 - l^6T^3)$

TABLE 5. Galois representations and Hecke polynomials for Eisenstein classes. See Sections 3 and 4 for explanation of notation.

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